MATH20132 Calculus of Several Variable. 2020-21

Problems 10 Differential forms

1. Given these 1-forms $\boldsymbol{\omega}$ evaluate $\boldsymbol{\omega}_{\mathbf{a}}(\mathbf{t})$ at the given $\mathbf{a}$ and t.

i. $\boldsymbol{\omega} = (x^2 + y^2)\, dx + xydy$ at $\mathbf{a} = (1, -1)^T$ and $\mathbf{t} = (2, -1)^T$.

ii. $\boldsymbol{\omega} = 3dx + 4dy$ at

a. $\mathbf{a} = (1, -1)^T$ and $\mathbf{t} = (2, -1)^T$,

b. $\mathbf{a} = (2, 3)^T$ and $\mathbf{t} = (2, -1)^T$

2. i. Find the differential of each of the following functions as 1-forms, $\boldsymbol{\omega} : \mathbb{R}^n \to \mathrm{Hom}(\mathbb{R}^n, \mathbb{R})$, with the appropriate n.

a. $f(\mathbf{x}) = x \sin(x^2 y) + y$ for $\mathbf{x} \in \mathbb{R}^2$,

b. $g(\mathbf{x}) = x^4 - 3x^2y^2 + yz^2$ for $\mathbf{x} \in \mathbb{R}^3$.

ii. a. In Part i.a calculate $\boldsymbol{\omega}_{\mathbf{a}}(\mathbf{t})$ with $\mathbf{a} = (2, -3)^T$ and $\mathbf{t} = (5, -2)^T$.

b. In Part i.b calculate $\boldsymbol{\omega}_{\mathbf{a}}(\mathbf{t})$ with $\mathbf{a} = (2, -3, 1)^T$ and $\mathbf{t} = (5, -2, 4)^T$.

3. In each of the following parts can you find a function $f : \mathbb{R}^2 \to \mathbb{R}$ such that

i. $df = (x^2 + y^2)\, dx + 2xydy$,

ii. $df = (1 + e^x)\, dy + e^x (y - x)\, dx$,

iii. $df = e^y dx + x(e^y + 1)\, dy$.

Give your reasons and, if the function exists, write it out.

Idea Recall that if f is Fréchet differentiable then

$$df = \frac{\partial f}{\partial x} dx + \frac{\partial f}{\partial y} dy + ...$$

Given a form $g = g^1 dx + g^2 dy + ...$ assume there exists $f : df = g$. This means $\partial f / \partial x = g^1$.

Integrate w.r.t x so $f = \int g + C$ where C depends on all variables *other than* x.

Differentiate w.r.t. y when we must have $\partial \left(\int g + C\right)/\partial y = g^2$.

Integrate w.r.t. y and continue, next differentiating w.r.t the third variable. Either this process will work and you construct f, or you obtain a contradiction and conclude that no such f exists.

4. *In the lectures we showed that if a* 1*-form* $\boldsymbol{\omega}$ *is exact, i.e.* $\exists f : df = \boldsymbol{\omega}$, *then it is closed, i.e.* $\partial\omega_i/\partial x^j = \partial\omega_j/\partial x^i$ *for all pairs* (i, j). *I stated that the converse is not true, i.e. not all closed forms are exact. In brief*

$$\boxed{\begin{array}{rcl} \text{exact} & \Longrightarrow & \text{closed} \\ \text{closed} & \not\Longrightarrow & \text{exact.} \end{array}}$$

In each of the following, determine whether the 1-form ω is closed, and if closed, exact. If exact, find all functions f such that $df = \boldsymbol{\omega}$:

i. $\boldsymbol{\omega} = y\,dx : \mathbb{R}^2 \to \mathrm{Hom}(\mathbb{R}^2, \mathbb{R})$;

ii. $\boldsymbol{\omega} = xy\,dx + (x^2/2)\,dy : \mathbb{R}^2 \to \mathrm{Hom}(\mathbb{R}^2, \mathbb{R})$;

iii. $\boldsymbol{\omega} = 2xydx + (x^2 + 4yz)\,dy + 2y^2dz : \mathbb{R}^3 \to \mathrm{Hom}(\mathbb{R}^3, \mathbb{R})$;

iv. $\boldsymbol{\omega} = x\,dx + xz\,dy + xy\,dz : \mathbb{R}^3 \to \mathrm{Hom}(\mathbb{R}^3, \mathbb{R})$.

5. Let $\boldsymbol{\omega} : U \subseteq \mathbb{R} \to \mathrm{Hom}(\mathbb{R}, \mathbb{R})$ be a 1-form on $\mathbb{R}$. This means there exists $f : U \to \mathbb{R}$ such that $\boldsymbol{\omega} = f dx$. Let $\boldsymbol{\gamma}$ be the closed interval $[a, b] \subset \mathbb{R}$ directed from a to b. Prove that

$$\int_{\gamma} \boldsymbol{\omega} = \int_a^b f(x)dx.$$

This is saying that for 1*-forms on* $\mathbb{R}$ *the integral along a line given in the lectures reduces to the previous definition of integration known from School days.*

Hint What parametrisation $g : [a, b] \to \boldsymbol{\gamma}$ should be chosen?

6. Integrate the following 1-forms on the curves given.

i. $\boldsymbol{\omega} = (xz + y)\,dx + z^2dy + xydz$ over the curve $\boldsymbol{\gamma}$ parametrised by $\mathbf{g}(t) = (t, t^2, 1+t)^T, \quad 0 \le t \le 2,$

ii. $\boldsymbol{\omega} = yzdx - xdy - (y - z)\,dz$ over the curve $\boldsymbol{\gamma}$ parametrised by $\mathbf{g}(t) = (t^2, t-1, t+1)^T$, $0 \le t \le 1$.

7. Integrate the 1-form $\boldsymbol{\omega} = ydx + xydy$ on $\mathbb{R}^2$ around the closed curve $\boldsymbol{\gamma} : x^2 + y^2 = R^2$, for a fixed R, in a counter-clockwise direction.

Hint Parametrise the curve by

$$\mathbf{g}(t) = \begin{pmatrix} R\cos t \\ R\sin t \end{pmatrix}$$

for $0 \le t \le 2\pi$. For the final integration it may save time to recall that $\int_0^{2\pi} \sin^2 t dt = \pi$.

8. i. Prove that the 1-form

$$\boldsymbol{\omega} = \frac{-y}{x^2+y^2}dx + \frac{x}{x^2+y^2}dy : \mathbb{R}^2 \setminus \{\mathbf{0}\} \to \mathrm{Hom}(\mathbb{R}^2, \mathbb{R})$$

is a closed form.

ii. Let $\boldsymbol{\gamma}$ be the unit circle centre $\mathbf{0}$ in $\mathbb{R}^2$. Evaluate $\int_{\boldsymbol{\gamma}} \omega$.

iii. Deduce that ω is not exact.

This is an illustration of the result

closed $\not\Rightarrow$ exact.

9. i. Integrate the 1-form $\boldsymbol{\omega} = (x - z)\,dx + xyzdy + (z - y)\,dz$ along a closed path $\Gamma = \boldsymbol{\gamma}_1 \cup \boldsymbol{\gamma}_2 \cup \boldsymbol{\gamma}_3 \cup \boldsymbol{\gamma}_4$ of four parts, each parametrised by:

- $\mathbf{g}_1(s) = (s, 0, s)^T$ for s from 0 to 1;
- $\mathbf{g}_2(t) = (1+t, t, 1)^T$ for t from 0 to 2;
- $\mathbf{g}_3(s) = (s+2, 2s, s)^T$ for s from 1 to 0 (note the direction of s);

- $\mathbf{g}_4(t) = (t, 0, 0)^T$ for t from 2 to 0.

ii. Prove that the form $\boldsymbol{\omega}$ is not exact.

10. Evaluate the 2-form

$$\left(x^2yzdx \wedge dy + (x - z)\, dx \wedge dz + yzdy \wedge dz\right)_{\mathbf{a}} (\mathbf{v}_1, \mathbf{v}_2)$$

where $\mathbf{a} = (1, -1, 2)^T$ and $\mathbf{v}_1 = (1, 2, 3)^T$, $\mathbf{v}_2 = (4, -5, 3)^T$.

11. Integrate the 2-form $\boldsymbol{\beta} = yzdx \wedge dy + dx \wedge dz - (xy + 1)\, dy \wedge dz$ over the surface

$$\mathcal{R} = \left\{ \begin{pmatrix} s + t \\ st \\ s \end{pmatrix} : 0 \le s \le 1, 0 \le t \le 2 \right\}.$$

12 Integrate the 2-form $\boldsymbol{\beta} = (y - 1)\, dx \wedge dy$ over the region $\mathcal{D}(R) : x^2 + y^2 \le R^2$ for fixed R.

Hint Parametrise the region by

$$\mathbf{g}(\mathbf{t}) = \begin{pmatrix} r\cos\theta \\ r\sin\theta \end{pmatrix},$$

where $\mathbf{t} = (r, \theta)$ with $0 \le r \le R$ and $0 \le \theta \le 2\pi$.

13. Let $\boldsymbol{\alpha}$, $\boldsymbol{\beta}$ and $\boldsymbol{\gamma}$ be 1 - forms given as

$\boldsymbol{\alpha} = x\, dx + yz\, dy + xyz\, dz,$

$\boldsymbol{\beta} = y^2\, dx + z\, dy - 3(x - 1)\, dz$ and

$\boldsymbol{\gamma} = z\, dx \wedge dy - y\, dx \wedge dz + x\, dy \wedge dz.$

Find $\boldsymbol{\alpha} \wedge \boldsymbol{\alpha}$, $\boldsymbol{\alpha} \wedge \boldsymbol{\beta}$ and $\boldsymbol{\alpha} \wedge \boldsymbol{\gamma}$.

14 Find the derivatives of

i. $ydx + xydy$ (seen in Question 7),

ii. $(x - z)\, dx + xyzdy + (z - y)\, dz$ (seen in Question 9),

Have you seen your answers in other questions on this sheet. If so, what conclusions can you draw?

Hint Think about Stokes' Theorem, surfaces and boundaries.

15 For the forms in Question 13, find $d\boldsymbol{\alpha}$, $d\boldsymbol{\beta}$ and $d\boldsymbol{\gamma}$.

Additional Questions

16. Integrate the 1-form $\boldsymbol{\omega} = yxdy$ along the boundary of the ellipse

$$\frac{(x-1)^2}{4} + \frac{(y+2)^2}{9} = 1,$$

in the counter-clockwise direction.

Hint to parametrise this curve use the fact that $\cos^2 t + \sin^2 t = 1$. For the final integration it may save time to note that $\int_0^{2\pi} \cos^2 t dt = \pi$.

17. Integrate the 1-form $\boldsymbol{\omega} = (x + y + z)\, dx + y^2 dy + xydz$ along $\boldsymbol{\gamma}$, the boundary of the unit circle in the x - y plane, centre the origin, in the counter-clockwise direction.

Hint Even though the circle lies in the x - y plane the 1-form is defined on $\mathbb{R}^3$ and so you have to parametrise the circle in $\mathbb{R}^3$.

18. Integrate the 2-form $\boldsymbol{\beta} = -dx \wedge dy + (y-1)\, dx \wedge dz + xdy \wedge dz$ over $\mathcal{H}$, the upper half of the unit sphere, so $x^2 + y^2 + z^2 = 1$ with $z \geq 0$.

Hint Parametrise this surface by the spherical coordinates

$$\mathbf{g}(\mathbf{t}) = \begin{pmatrix} \sin\phi\cos\theta \\ \sin\phi\sin\theta \\ \cos\phi \end{pmatrix},$$

where $\mathbf{t} = (\phi, \theta)$, with $0 \leq \phi \leq \pi/2$ and $0 \leq \theta \leq 2\pi$.

19. Integrate the 2-form $\boldsymbol{\omega} = (x^2 y + y^2 z^2)\, dx \wedge dy + y^3 z dx \wedge dz + xy^2 z dy \wedge dz$ over the surface of the sphere $x^2 + y^2 + z^2 = a$.

20. Integrate the 2-form $\boldsymbol{\beta} = -dx \wedge dy + (y-1)\, dx \wedge dz + xdy \wedge dz$ over $\mathcal{D}$, the region $x^2 + y^2 \leq 1$ in the x - y plane.

Hint As in question 17, though the region of integration lies in the x - y plane the form is defined on $\mathbb{R}^3$ and so you have to choose a parametrisation of the region as a subset of $\mathbb{R}^3$.

21. Explain why Questions 17, 18 and 20 together illustrate Stoke's Theorem.

22. Integrate the form $\boldsymbol{\beta} = y dx \wedge dy$ over the area within the ellipse

$$\frac{(x-1)^2}{4} + \frac{(y+2)^2}{9} = 1.$$

Hint parametrise this region by

$$\mathbf{g}(\mathbf{t}) = \begin{pmatrix} 1 + 2r\cos t \\ -2 + 3r\sin t \end{pmatrix}$$

where $\mathbf{t} = (r,t)^T$ satisfies $0 \leq r \leq 1, 0 \leq t \leq 2\pi$.

Note this question is related to Question 16 by Stoke's Theorem.

If you have been reading the asides in my notes on Vector Calculus the following may be of interest.

23. Suppose that $\mathbf{f}$, $\mathbf{g} : \mathbb{R}^3 \to \mathbb{R}^3$ are two vector fields on $\mathbb{R}^3$. Recall, from the asides in the notes, the vectors

$$d\mathbf{r} = \begin{pmatrix} dx^1 \\ dx^2 \\ \vdots \\ dx^n \end{pmatrix} \quad \text{and} \quad \mathbf{n} = \begin{pmatrix} dy \wedge dz \\ dz \wedge dx \\ dx \wedge dy \end{pmatrix}.$$

Prove that

$$(\mathbf{f} \bullet d\mathbf{r}) \wedge (\mathbf{g} \bullet d\mathbf{r}) = \mathbf{f} \times \mathbf{g} \bullet \mathbf{n}.$$

We say that $\mathbf{f} \bullet d\mathbf{r}$ and $\mathbf{g} \bullet d\mathbf{r}$ are the 1-forms associated with $\mathbf{f}$ and $\mathbf{g}$ while $\mathbf{f} \times \mathbf{g} \bullet \mathbf{n}$ is the 2-form associated with $\mathbf{f} \times \mathbf{g}$. Hence this result says that the wedge product of the 1-forms associated with $\mathbf{f}$ and $\mathbf{g}$ is the 2-form associated with the vector product of $\mathbf{f}$ and $\mathbf{g}$.